MCMC Fall 2025
Advanced Answer Key

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1 Speed

1. 0
2. 168
3. 0
4. 1000
5. $2\sqrt{314}$
6. $\frac{288}{455}$
7. $\frac{15}{2}$
8. $\frac{7411}{9520}$
9. $\pi - 2$
10. 1800
11. $2\sqrt{5}$
12. 17
13. 7
14. 3439
15. 2
16. 995
17. 1
18. 40000
19. 6
20. 0

2 Accuracy

1. A
2. C
3. A
4. E
5. D
6. B
7. A
8. A
9. C
10. C
11. B
12. D
13. B
14. E
15. A
16. E
17. B
18. B
19. C
20. C

Free Response Questions: Introduction to Graph Theory

Definitions in Graph Theory

Graph: A graph G is defined as an ordered pair $G = (V, E)$, where V is a set of vertices (or nodes) and E is a set of edges (or links), which are two-element subsets of V.

Simple Graph: A simple graph is a graph in which all edges are undirected (the edge (u, v) is the same as (v, u)) and there are no loops (edges that connect a vertex to itself).

Adjacent: Two vertices u and v are considered adjacent if they are joined by an edge.

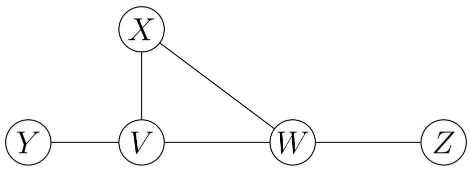
Degree of a Vertex: The degree of a vertex v, denoted $d(v)$, is the number of vertices adjacent to it.

Path: A path is a sequence of edges that allows you to travel from a starting vertex u to an ending vertex v. The length of a path is the number of edges it contains.

Connected Graph: A graph is connected if for every pair of vertices (u, v) in the graph, there exists a path between them.

Problems

Consider the graph G below for parts (a) and (b).



(a) Find the degree $d(v)$ for each vertex in the graph G.

$$\begin{aligned}
d(V) &= 3 && \text{(connected to } W\text{, } X\text{, and } Y\text{)} \\
d(W) &= 3 && \text{(connected to } V\text{, } X\text{, and } Z\text{)} \\
d(X) &= 2 && \text{(connected to } V \text{ and } W\text{)} \\
d(Y) &= 1 && \text{(connected only to } V\text{)} \\
d(Z) &= 1 && \text{(connected only to } W\text{)}
\end{aligned}$$

(b) Is graph G connected? Explain your reasoning.

Yes, graph G is connected. For every pair of vertices (u, v), there exists a path connecting them. For example, even though Y is only adjacent to V and Z is only adjacent to W, we can travel from Y to Z via the path $Y \to V \to W \to Z$. Thus, the graph is connected.

(c) Let H be a simple graph where every vertex has degree k, with $k \geq 2$. Prove that H must contain a path of length at least k.

Proof: Let H be a k-regular graph, meaning every vertex has degree k. Choose any vertex v_0. Starting from v_0, since it has k neighbors, we can construct a path by successively visiting new vertices. Because each vertex has degree $k \geq 2$, we can always continue the path until no new vertex remains unvisited. By the handshaking lemma, H has at least $k + 1$ vertices. Therefore, the graph contains a path of length at least k.

$\square$